JOURNAL OF COMBINATORIAL THEORY, Series B 35, 83–92 (1983)

Crossing-Critical Edges and Kuratowski Subgraphs of a Graph

JOZEF ŠIRÁŇ

Department of Telecommunications, EF SVST, Vazovova 5, Technical University, Bratislava 81219 Czechoslovakia

Communicated by the Editors

Received January 12, 1983

An edge e of a graph G is said to be crossing-critical if $\mathrm{cr}(G-e) < \mathrm{cr}(G)$, where $\mathrm{cr}(G)$ denotes the crossing number of G on the plane. It is proved that any crossing-critical edge e of a graph G for which $\mathrm{cr}(G-e) \leqslant 1$ belongs to a subdivision of K_5 or $K_{3,3}$, the Kuratowski subgraphs of G. Further, as regards crossing-critical edges e of G for which $\mathrm{cr}(G-e) \geqslant 5$, it is shown that the properties of "being a crossing-critical edge of G" and "being contained in a Kuratowski subgraph of G" are independent.

1. INTRODUCTION AND PRELIMINARIES

An edge e of a graph G is said to be crossing-critical if $\mathrm{cr}(G-e) < \mathrm{cr}(G)$, where $\mathrm{cr}(G)$ denotes the crossing number of G on the plane. Any subdivision of K_5 or $K_{3,3}$ in G is called a Kuratowski subgraph of G.

In 1954, Dirac and Schuster [1] published the first relatively simple proof of Kuratowski's theorem on planar graphs [6]. Their proof is based on a description of a minimal counterexample, i.e., non-planar graph G_0 with minimum number of vertices and edges such that G_0 contains no Kuratowski subgraphs. Of course, each edge of G_0 must be crossing-critical and $\mathrm{cr}(G_0 - e) = 0$. The crucial step was to show that one of these crossing-critical edges belongs to a Kuratowski subgraph of G_0. In fact, they thereby proved that any crossing-critical edge e of an arbitrary graph G satisfying $\mathrm{cr}(G-e) = 0$ belongs to a Kuratowski subgraph of G.

The purpose of this paper is to generalize this result of Dirac and Schuster. In particular, we will show that any crossing-critical edge e of a graph G for which $\mathrm{cr}(G-e) \leqslant 1$ belongs to a Kuratowski subgraph of G (Section 2). Attempts to extend the last result to all crossing-critical edges of a graph G are hopeless even when the difference $\mathrm{cr}(G) - \mathrm{cr}(G-e)$ is arbitrarily large. In Section 3 we give an example of a graph G containing a crossing-critical edge e with $\mathrm{cr}(G-e) = 5$ and such that e belongs to no

0095-8956/83 $3.00

Kuratowski subgraph of G. Moreover, for any number n we construct a simple 3-connected graph G containing an edge e such that $\mathrm{cr}(G) - \mathrm{cr}(G-e) \geqslant n$ while e belongs to no Kuratowski subgraph of G.

We will consider only finite, undirected graphs without loops, i.e., we allow multiple edges. Graphs containing only edges of multiplicity one are called simple. The remaining terminology and notation is essentially the same as Harary [4], except as indicated.

For the concepts of a drawing and an optimal drawing of a graph G on the plane see, e.g., [2]. If D is a drawing of G on the plane, we shall use the same notation for vertices (edges) of G and corresponding nodes (arcs) of D.

Let vertices u, v of G be joined in G by an edge of multiplicity m. For a natural number λ denote by $G + \lambda uv$ ($G - \lambda uv$ for $\lambda \leqslant m$) the graph obtained from G by adding (removing) λ edges joining u, v. If $\lambda = 1$ we simply write $G + uv$ ($G - uv$).

Before concluding this section let us recall some relevant definitions and results from [8, 9] which we will use later. If S is a subdivision of $K_{3,3}$, two branch vertices (i.e., vertices of degree $\geqslant 3$) u, v of S are said to be independent if any $u-v$ path in S contains at least one branch vertex of S different from u, v.

Proposition A [9, Lemma 2]. *Let $e = uv$ be an edge of a 3-connected non-planar graph G such that e belongs to no Kuratowski subgraph of G. Then u, v are independent branch vertices of any subdivision of $K_{3,3}$ in G.*

Proposition B [9, Theorem 2]. *Any edge of a 4-connected non-planar graph G belongs to a Kuratowski subgraph of G.*

Let H, K be subgraphs of a graph G. We say that (H, K) is a (u, v)-decomposition of G if the following holds:

(a) H and K have precisely two common vertices u, v;

(b) each edge of G belongs to exactly one of the subgraphs H, K.

Proposition C [8, Theorem 1]. *Let (H, K) be a (u, v)-decomposition of G. Suppose that $\mathrm{cr}(H + uv) = \mathrm{cr}(H)$. Let $\lambda = \lambda_H(u, v)$ denote the local edge-connectivity of H with respect to vertices u, v. Put $K^* = K + \lambda uv$. Then*

$$\mathrm{cr}(G) = \mathrm{cr}(H) + \mathrm{cr}(K^*).$$

2. A Generalization of the Result of Dirac and Schuster

Recall that Dirac and Schuster [1] actually proved the following theorem which is easily seen to be equivalent to Kuratowski's theorem on planar graphs [6].

THEOREM 1. *Let e be a crossing-critical edge of a graph G such that $\mathrm{cr}(G-e)=0$. Then e belongs to a Kuratowski subgraph of G.*

There are several ways to prove the following Theorem 2 which generalizes Theorem 1. The proof published here utilizes ideas of Thomassen's short proof of Kuratowski's theorem [10].

THEOREM 2. *Let e be a crossing-critical edge of a graph G for which $\mathrm{cr}(G-e)\leqslant 1$. Then e belongs to a Kuratowski subgraph of G.*

Proof. Let G be a graph with minimum number of vertices containing a crossing-critical edge e such that $\mathrm{cr}(G-e)\leqslant 1$ while e belongs to no Kuratowski subgraph of G. Obviously, G is 2-connected and Theorem 1 implies that $\mathrm{cr}(G-e)=1$. To investigate some properties of our minimum counterexample in detail we prove a series of six lemmas.

LEMMA 1. *The vertex connectivity $\kappa(G)=3$.*

Proof of Lemma 1. It has already been proved that $\kappa(G)\geqslant 2$. The inequality $\kappa(G)\leqslant 3$ is a consequence of Proposition B. It remains to show that $\kappa(G)>2$. Assume the contrary and let vertices u, v form a cut set of G. Then there exist two subgraphs H, K of G such that (H,K) is a (u,v)-decomposition of G. We may suppose that e belongs to K.

Consider the graph $H+uv$. If $\mathrm{cr}(H+uv)>\mathrm{cr}(H)$, then the new edge uv belongs to a Kuratowski subgraph of $H+uv$ since $\mathrm{cr}(H)\leqslant 1$ and H has fewer vertices than G. But in this case any edge of K (in particular, e) belongs to a Kuratowski subgraph of G—a contradiction. Therefore $\mathrm{cr}(H+uv)=\mathrm{cr}(H)$.

Let $\lambda=\lambda_H(u,v)$ denote the local edge-connectivity of H with respect to vertices u,v. Put $K^*=K+\lambda uv$. According to Proposition C, $\mathrm{cr}(G)=\mathrm{cr}(H)+\mathrm{cr}(K^*)$ and $\mathrm{cr}(G-e)=\mathrm{cr}(H)+\mathrm{cr}(K^*-e)$. Combining the last two equalities with our assumptions we obtain $\mathrm{cr}(K^*-e)<\mathrm{cr}(K^*)$ and $\mathrm{cr}(K^*-e)\leqslant 1$, which contradicts the minimality of G. Lemma 1 follows.

Let G/e denote the graph obtained from G by contracting the edge e (cf. [10]).

LEMMA 2. *The graph G/e is planar.*

Proof of Lemma 2. According to a theorem of Hall [3] G contains a subdivision of $K_{3,3}$. Our Proposition A implies that e is an edge joining two independent branch vertices of each subdivision of $K_{3,3}$ in G, whence G/e cannot contain any subdivision of $K_{3,3}$. Suppose that G/e contains a subdivision of K_5. It is easy to show that, in an arbitrary 2-connected graph L containing a subdivision of K_5, each edge of L belongs to a Kuratowski subgraph of L. Thus, G contains no subdivision of K_5, and we deduce that there is a subdivision S of K_5 in G/e containing the vertex obtained by contracting the edge e. But starting from S it is easy to find a Kuratowski subgraph of G which contains the edge e. This contradiction shows that G/e contains no Kuratowski subgraphs, i.e., G/e is planar.

Denote by s, t the vertices of G incident with the edge e, and w the vertex of G/e obtained by identifying vertices s and t.

LEMMA 3. $\kappa(G/e)=2$ *and* $\kappa(G/e-w)=1$.

Proof of Lemma 3. From Lemma 1 we deduce that $\kappa(G/e)\geqslant 2$ and $\kappa(G/e-w)\geqslant 1$. Suppose $\kappa(G/e-w)\geqslant 2$. Since G/e is planar, it can be proved by exactly the same method as in [10, proof of Theorem 3.2] that e belongs to a Kuratowski subgraph of G, a contradiction. Therefore $\kappa(G/e-w)=1$, which implies $\kappa(G/e)=2$. Q.E.D.

Consider a fixed plane representation (see [4]) Γ of the graph G/e such that the vertex w is contained in the boundary of the outer face of Γ. The representation Γ induces a plane representation $\Gamma_w \subseteq \Gamma$ of the vertex-deleted graph $H=G/e-w$. Denote by $\mathrm{bc}(H)$ the block-cutpoint tree of the graph H (cf. [4]). It follows from Lemma 3 that $\mathrm{bc}(H)$ is a non-trivial tree, and either it is a snake or it contains a subdivision S of $K_{1,3}$. In the second case, the branch vertex of S corresponds either to a block or to a cut vertex of H. Each of these three possibilities will be handled separately.

LEMMA 4. *Let* $\mathrm{bc}(H)$ *be a snake. Then* $G-e$ *contains the subgraph* H_1 *depicted in Fig.* 1. *Moreover, if the set* $\{s,t,r_i\}$ *separates vertices* w_j *and* w_k (*for some* i, j, k) *in* H_1, *then the same holds in* $G-e$.

Proof of Lemma 4. Let $B_1, B_2,\ldots, B_k$ be the family of all blocks of Γ_w such that B_j and B_{j+1} have exactly one cut vertex c_j of Γ_w in common; thus $c_1, c_2,\ldots, c_{k-1}$ are all the cut vertices of Γ_w. Denote by C_j the boundary of B_j, i.e., $C_1\cup C_2\cup\cdots\cup C_k$ is the boundary of the outer face of Γ_w.

Since G is 3-connected, each of vertices s, t is adjacent to at least one vertex of B_1 different from c_1 and to at least one vertex of B_k different from c_{k-1}. Consider a block B_j, $2\leqslant j\leqslant k-1$. Suppose $B_j\neq K_2$ and let P_j, Q_j be components of $C_j-\{c_{j-1}, c_j\}$. If there are two different vertices s_1, $t_1\in P_j$ (or, symmetrically, s_1, $t_1\in Q_j$) adjacent to s, t respectively, then the edge

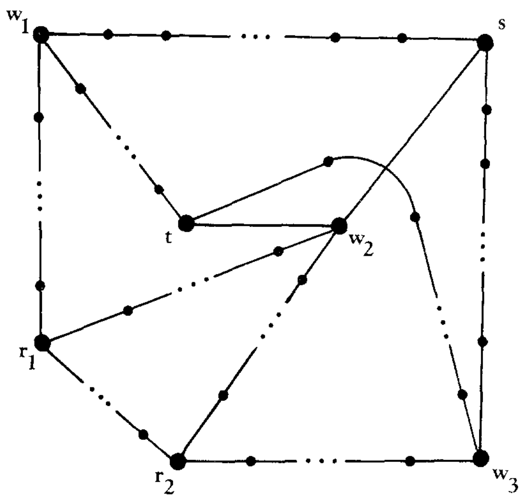

FIG. 1. The graph H_1. (Note the graph $H_1 + st$, although contractible to K_5, contains an edge (st) which belongs to no Kuratowski subgraph of $H_1 + st$.)

$e = st$ would clearly belong to a subdivision of $K_{3,3}$ in G (compare [10, proof of Theorem 3.2]). On the other hand, if there are vertices $s_1 \in P_j$, $s_2 \in Q_j$ both adjacent to s (or, t) in G we can easily find a subdivision of $K_{3,3}$ in G containing e; it suffices to consider the subgraph of G spanned by the cycle C_j, edges ss_1, ss_2, st, and paths $t - c_{j-1}$ and $t - c_j$ containing no edge of B_j (the case when s is replaced by t can be handled similarly). From all these facts we deduce that either (1) $P_j \cup Q_j$ contains exactly one vertex adjacent to both s, t, or we may suppose the notation to be chosen in such way that (2) s is adjacent to no vertex of Q_j and t is adjacent to no vertex of P_j.

Now, assume that (2) holds for each $B_j \neq K_2$, $2 \leqslant j \leqslant k-1$. Consider the blocks B_1 and B_k. If $B_1 \neq K_2$, by the same arguments as above we can show that there is a vertex v_1 of C_1 different from c_1 such that, using the notation P_1, Q_1 for the components of $C_1 - \{c_1, v_1\}$, s is adjacent to no vertex of Q_1 while t is adjacent to no vertex of P_1; a similar result holds for B_k. But then we can change Γ_w (by "switching" P_j, Q_j if necessary) to a representation of H which can be extended to a plane representation of the whole graph G, a contradiction.

Thus, there is a $B_j \neq K_2$, $2 \leqslant j \leqslant k-1$ for which (1) holds. Let $w_2 \in C_j - \{c_{j-1}, c_j\}$ be the vertex adjacent to both s, t. Choose vertices $w_1 \in B_1$, $w_3 \in B_k$ both adjacent to s and such that $w_1 \neq c_1$, $w_3 \neq c_{k-1}$. Put $r_1 = c_{j-1}$, $r_2 = c_j$. One can see that the cycle C_j together with $r_1 - w_1$ and $r_2 - w_3$ paths in Γ_w and edges or paths of type $w_i - s$, $w_i - t$ induce the subgraph H_1 of $G - e$ depicted in Fig. 1 with the required properties. The proof of Lemma 4 is complete.

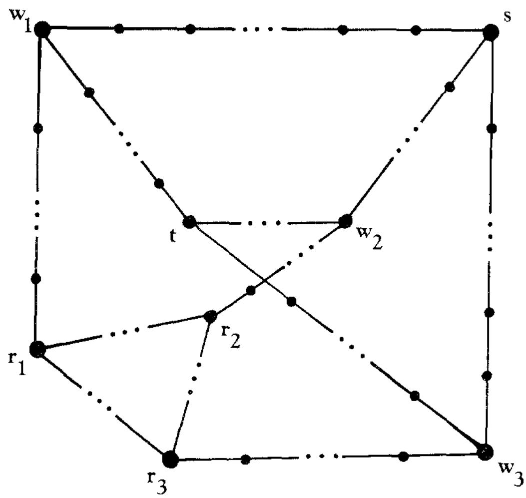

FIG. 2. The graph H_2.

LEMMA 5. *Let* bc(H) *contain a subdivision* S *of* $K_{1,3}$ *such that the branch vertex of* S *corresponds to a block of* H. *Then* $G-e$ *contains the subgraph* H_2 *of Fig.* 2. *Moreover, if the set* $\{s, t, r_i\}$ *separates vertices* w_j *and* w_k *in* H_2, *then the same holds in* $G-e$.

Proof of Lemma 5. We may suppose that each endvertex of S is also an endvertex of bc(H). Denote by B_1, B_2, B_3 the blocks of $H \cong \Gamma_w$ corresponding to the endvertices of S. Let B be the block of Γ_w represented by the branch vertex of S and $r_1, r_2, r_3 \in B$ be cut vertices of Γ_w such that any $B_i - B$ path in Γ_w passes through r_i, $1 \leqslant i \leqslant 3$. In each B_i choose a vertex w_i (adjacent to s) which is not a cut vertex of Γ_w. The required subgraph H_2 is now easily put together using the boundary cycle of B, paths $r_i - w_i$ (in Γ_w), $t - w_i$, and edges sw_i. Lemma 5 follows.

The last auxiliary result can be proved using similar arguments as above.

LEMMA 6. *Let* bc(H) *contain a subdivision* S *of* $K_{1,3}$ *such that the branch vertex of* S *represents a cut vertex* r_1 *of* H. *Then* $G-e$ *contains the subgraph* H_3 (*Fig.* 3). *Moreover, the set* $\{s, t, r_1\}$ *separates any two* w_i*'s in* $G-e$.

Now we are armed enough to complete the proof of Theorem 2. Let D be an optimal drawing of the graph $G-e$ on the plane. Topologically, this drawing (as a point-set) defines a decomposition of $\mathbb{R}^2 - D$ into a finite number of regions. Since e is crossing-critical, s, t cannot be on the boundary of the same such region. This implies that there is a simple closed

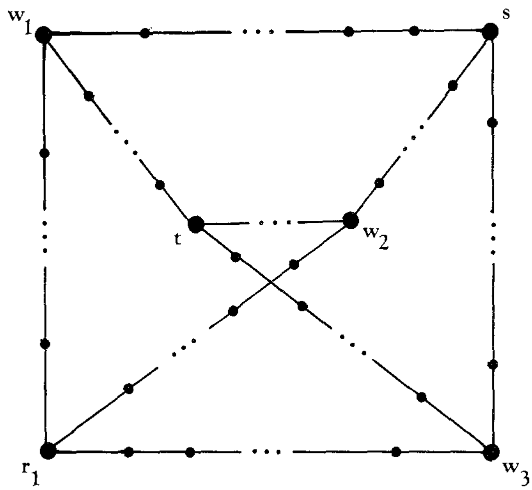

FIG. 3. The graph H_3.

topological curve $C \subseteq D$ such that, without loss of generality, s lies inside C while t lies outside C.

According to Lemmas 4–6, there is a k, $1 \leqslant k \leqslant 3$ such that D contains a drawing D_k of the subgraph H_k of $G-e$ (Figs. 1–3). Observe that D_3 is a subdivision of $K_{3,3}$, and for $k=1, 2$ it suffices to remove one path from D_k to obtain a subdivision of $K_{3,3}$. From the fact that $\mathrm{cr}(H_k)=\mathrm{cr}(G-e)=1$ it immediately follows that the curve C consists of arcs of D and, possibly, of two parts of arcs creating the unique crossing point of D_k. Considering all drawings of $K_{3,3}$ [5] and extending them to all possible drawings of $C \cup D_k$ one can check that there is always a (graph-theoretical) path P in $C \cup D_k$ avoiding r_i, s, t such that the endnodes of P are contained in the set $\{w_1, w_2, w_3\}$. However, Lemmas 4–6 imply that any two different w_j's in D can be separated by a set of the form r_i, s, t for suitable r_i. This final contradiction completes the proof of Theorem 2.

3. SOME COUNTEREXAMPLES

In this section we show that if there are further generalizations of Theorem 2 for crossing-critical edges e with $\mathrm{cr}(G-e) \leqslant n$, then $n \leqslant 4$.

THEOREM 3. *There is a graph G containing a crossing-critical edge e such that* $\mathrm{cr}(G-e)=5$ *and e belongs to no Kuratowski subgraph of G.*

Proof. Let G be the graph depicted in Fig. 4. Consider an optimal

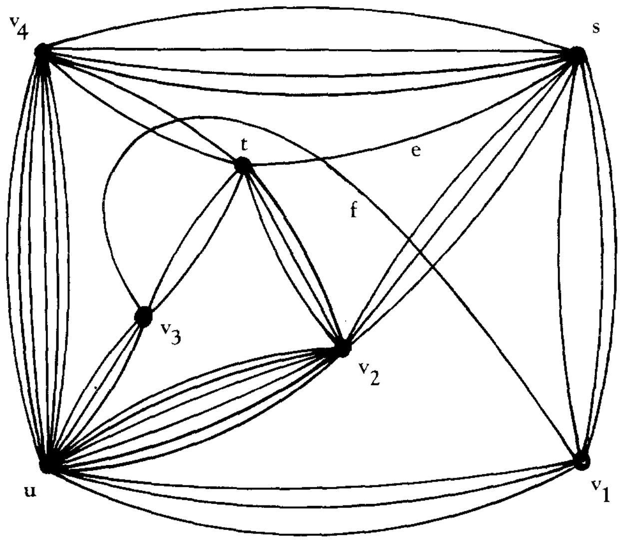

FIG. 4. The graph G.

drawing D of G on the plane. We may suppose that all arcs joining two arbitrary nodes a, b of D are drawn close one to another since D is optimal. Thus, we can speak, for example, about mn crossings created by arcs of multiplicities m and n. Looking at Fig. 4 we see that $\operatorname{cr}(G) \leqslant 6$ and $\operatorname{cr}(G-e) \leqslant 5$. We therefore assume that no arc of multiplicity 6 is crossed in D and arcs of multiplicity 2 do not cross those of multiplicity 3 in D.

Put $H = G - \{e, f\}$. From the facts above it follows that the drawing $D - \{e, f\}$ regarded as a subdrawing of D contains no crossing. Since H is planar and contains a spanning subgraph S such that S is a subdivision of a 3-connected planar graph, Whitney's theorem [12] implies that H has essentially only one imbedding on the sphere. It can be easily checked that, starting from this unique imbedding, there is only one way to complete it to a drawing of the whole graph G such that no arc of multiplicity 6 is crossed. In this case we obtain $\operatorname{cr}(G) \geqslant 6$ and $\operatorname{cr}(G-e) \geqslant 5$.

Thus, we have established that e is a crossing-critical edge and $\operatorname{cr}(G-e) = 5$. Moreover, one can see that e belongs to no Kuratowski subgraph of G. Theorem 3 follows.

COROLLARY 1. *For any* $n \geqslant 5$ *there is a (connected) graph* G_n *containing a crossing-critical edge* e *for which* $\operatorname{cr}(G_n - e) = n$ *and such that* e *belongs to no Kuratowski subgraph of* G_n.

The proof is obvious.

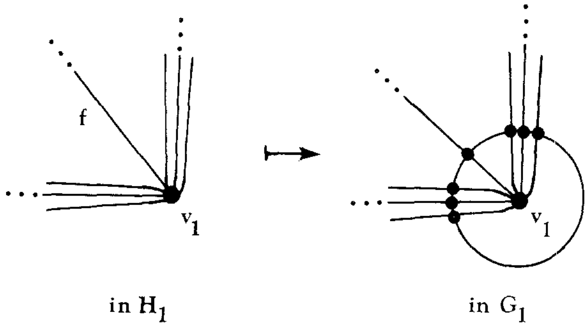

FIG. 5. The construction of the graph G_n for $n = 1$.

COROLLARY 2. *For any natural number n there is a simple* 3*-connected graph* G_n *containing an edge e for which* $\operatorname{cr}(G_n) - \operatorname{cr}(G_n - e) \geqslant n$ *and such that e belongs to no Kuratowski subgraph of* G_n.

Proof. Consider the graph G of Fig. 4. Replace each edge of G multiplicity m by an edge of multiplicity nm, obtaining a new graph H_n. Clearly, Theorem 3 implies that $\operatorname{cr}(H_n) = 6n^2$ and $\operatorname{cr}(H_n - st) \leqslant 6n^2 - n$.

Let D_n be an optimal drawing of the graph H_n obtained from the drawing D on Fig. 4 by adding arcs as indicated before. Now, let us modify the topological neighbourhoods of all vertices of D_n except for s, t, u by splitting the corresponding arcs and drawing a circuit connecting vertices of degree 2 close around vertices $v_1, \ldots, v_4$, as it is shown in Fig. 5 for the vertex v_1 in D_1.

By means of the above modification we obtain a new graph G_n from H_n. Obviously G_n is simple, 3-connected, $\operatorname{cr}(G_n) = 6n^2$ and $\operatorname{cr}(G_n - st) \leqslant 6n^2 - n$. With a little care we can check that again none of the edges joining vertices s, t in G_n belongs to a Kuratowski subgraph of G_n. This completes the proof of Corollary 2.

4. CONCLUDING REMARKS

Let A_n be the family of (mutually non-isomorphic) graphs G satisfying the following two conditions:

(a) $\operatorname{cr}(G) \geqslant n + 2$;

(b) $\operatorname{cr}(G - e) \leqslant \leqslant n$ for each edge e of G.

It is an easy consequence of Theorem 1 that A_0 is empty. However, an example due to Tomasta [11] shows that A_1 contains the product of 3-cycles $C_3 \times C_3$ which has crossing number 3 [7]. Theorem 2 implies that any graph of A_1 is covered by its Kuratowski subgraphs. Other than this, no more facts are known about A_1 (it is not clear even whether A_1 is finite or not).

Call an edge e of a graph G exceptional if e is crossing-critical and e belongs to no Kuratowski subgraph of G. It follows from [9, Theorems 1 and 2] that a simple 3-connected graph G can contain at most three exceptional edges (although we know no example of such a graph having more than one exceptional edge), while 4-connected graphs cannot contain exceptional edges at all. However, replacing the edge e of the graph G of Fig. 4 by a path we can see that graphs with connectivity at most 2 may contain any given number of exceptional edges.

REFERENCES

1. G. A. DIRAC AND S. SCHUSTER, A theorem of Kuratowski, *Indag. Math.* **16** (1954), 343–348.
2. P. ERDÖS AND R. K. GUY, Crossing number problems, *Amer. Math. Monthly* **80** (1973), 52–58.
3. D. W. HALL, A note on primative skew curves, *Bull. Amer. Math. Soc.* **49** (1943), 935–937.
4. F. HARARY, "Graph Theory," Addison-Wesley, Reading, Mass., 1969.
5. H. HARBORTH, Parity of numbers of crossings for complete n-partite graphs, *Math. Slovaca* **26** (2) (1976), 77–95.
6. K. KURATOWSKI, Sur le problème des courbes gauches en topologie, *Fund. Math.* **15** (1930), 271–283.
7. R. D. RINGEISEN AND L. W. BEINEKE, The crossing number of $C_3 \times C_n$, *J. Combin. Theory Ser. B* **24** (1978), 134–136.
8. J. ŠIRÁŇ, Additivity of the crossing number of graphs with connectivity 2, *Period. Math. Hungar.*, in press.
9. J. ŠIRÁŇ, Edges and Kuratowski sugraphs of non-planar graphs, *Math. Nachr.*, in press.
10. C. THOMASSEN, Planarity and duality of finite and infinite graphs, *J. Combin. Theory Ser. B* **29** (1980), 244–271.
11. P. TOMASTA, Personal communication.
12. H. WHITNEY, A set of topological invariants for graphs, *Amer. J. Math.* **55** (1933), 231–235.